GEOMETRIC CONVEXITY OF THE GENERALIZED SINE AND THE GENERALIZED HYPERBOLIC SINE

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Abstract. In the paper, the authors prove that the generalized sine function $\sin_{p,q}(x)$ and the generalized hyperbolic sine function $\sinh_{p,q}(x)$ are geometrically concave and geometrically convex, respectively. Consequently, the authors verify a conjecture posed in the paper "B. A. Bhayo and M. Vuorinen, *On generalized trigonometric functions with two parameters*, J. Approx. Theory **164** (2012), no. 10, 1415–1426; Available online at `http://dx.doi.org/10.1016/j.jat.2012.06.003`".

1. Introduction

It is well known from calculus that

$$\arcsin x=\int_0^x\frac{1}{(1-t^2)^{1/2}}\,\mathrm{d}t$$

for $0\le x\le 1$ and

$$\frac{\pi}{2}=\arcsin 1=\int_0^1\frac{1}{(1-t^2)^{1/2}}\,\mathrm{d}t.$$

We can define the sine function on $\left[0,\frac{\pi}{2}\right]$ as the inverse of the arcsine function and extend it to $(-\infty,\infty)$.

Let $1<p<\infty$. the arcsine can be generalized as

$$\arcsin_p x=\int_0^x\frac{1}{(1-t^p)^{1/p}}\,\mathrm{d}t,\quad 0\le x\le 1$$

and

$$\frac{\pi_p}{2}=\arcsin_p 1=\int_0^1\frac{1}{(1-t^p)^{1/p}}\,\mathrm{d}t.$$

The inverse of the function $\arcsin_p$ on $\left[0,\frac{\pi_p}{2}\right]$ is called the generalized sine function and denoted by $\sin_p$. By standard extending procedures as the sine function done, we may obtain a differentiable function on the whole real line $(-\infty,\infty)$, which coincides with the sine when $p=2$. It is easy to see that the function $\sin_p$ is strictly increasing and concave on $\left[0,\frac{\pi_p}{2}\right]$. Similarly, we can define the generalized cosine, tangent, hyperbolic functions and their inverses.

For $p,q>1$, let

$$F_{p,q}(x)=\int_0^x(1-t^q)^{-1/p}\,\mathrm{d}t,\quad x\in[0,1]$$

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and

$$\frac{\pi_{p,q}}{2} = \int_0^1 (1-t^q)^{-1/p}\,\mathrm{d}t.$$

Then $F_{p,q} : [0,1] \to \bigl[0, \frac{\pi_{p,q}}{2}\bigr]$ is an increasing homeomorphism. We denote this function $F_{p,q}$ by $\arcsin_{p,q}$. Thus, its inverse

$$\sin_{p,q} = F_{p,q}^{-1}$$

is defined on the interval $\bigl[0, \frac{\pi_{p,q}}{2}\bigr]$. By a similar extension to the sine function, we can find a differentiable function $\sin_{p,q}$ defined on $\mathbb{R}$. We call $\sin_{p,q}$ the generalized (p,q)-sine function.

We can also define

$$\arccos_{p,q} x = \arcsin_{p,q}\bigl[(1-x^p)^{1/q}\bigr],$$

see [2, 4], and the inverse of the generalized (p,q)-hyperbolic sine function

$$\operatorname{arcsinh}_{p,q}(x) = \int_0^x (1+t^q)^{-1/p}\,\mathrm{d}t, \quad x \in (0,\infty).$$

Their inverse functions are

$$\sin_{p,q} : \left(0, \frac{\pi_{p,q}}{2}\right) \to (0,1), \quad \cos_{p,q} : \left(0, \frac{\pi_{p,q}}{2}\right) \to (0,1),$$

and

$$\sinh_{p,q} : \bigl(0, m_{p,q}^*\bigr) \to (0,\infty),$$

where

$$m_{p,q}^* = \int_0^\infty (1+t^q)^{-1/p}\,\mathrm{d}t.$$

When $p = q$, the (p,q)-functions $\sin_{p,q}$, $\cos_{p,q}$, $\sinh_{p,q}$, $\arcsin_{p,q}$, $\arccos_{p,q}$, and $\operatorname{arcsinh}_{p,q}$ reduce to p-functions $\sin_p$, $\cos_p$, $\sinh_p$, $\arcsin_p$, $\arccos_p$, and $\operatorname{arcsinh}_p$ respectively. See [3, 5, 7]. In particular, when $p = q = 2$, the (p,q)-functions become our familiar trigonometric and hyperbolic functions.

Recently, the generalized trigonometric and hyperbolic functions, including (p,q)-functions and p-functions, have been studied by many mathematicians from different points of view. See [1, 6, 8, 11]. In [4], the authors gave basic properties of the generalized (p,q)-trigonometric functions. In [5], the authors generalized some classical inequalities for trigonometric and hyperbolic functions, such as Mitrinović-Adamović inequality, Lazarević's inequality, Huygens-type inequalities, and Wilker-type inequalities.

In [2], the authors found that the functions $\arcsin_{p,q}$ and $\operatorname{arcsinh}_{p,q}$ can be expressed in terms of Gaussian hypergeometric functions. Applying the vast available information about hypergeometric functions, some remarkable properties and inequalities for generalized trigonometric and hyperbolic functions are obtained. Moreover, they raised the following conjecture.

Conjecture 1.1 ([2, p. 1421, Conjecture 2.11])**.** *If $p,q \in (1,\infty)$ and $r,s \in (0,1)$, then*

$$\sin_{p,q}\sqrt{rs} \ge \sqrt{\sin_{p,q} r \sin_{p,q} s} \tag{1.1}$$

and

$$\sinh_{p,q}\sqrt{rs} \le \sqrt{\sinh_{p,q} r \sinh_{p,q} s}. \tag{1.2}$$

The main purpose of this paper is to discover the geometric convexity of $\sin_{p,q}(x)$ and $\sinh_{p,q}(x)$ and to give an affirmative answer to the above stated [2, p. 1421, Conjecture 2.11].

Our main results may be formulated in the following theorem.

Theorem 1.1. *Let $p, q \in (1, \infty)$ and $r, s, x \in (0, 1)$. Then*

(1) *the function $\sin_{p,q}(x)$ is geometrically concave function;*
(2) *the function $\sinh_{p,q}(x)$ is geometrically convex function.*

Corollary 1.1. *Let $p, q \in (1, \infty)$ and $r, s, x \in (0, 1)$. Then the inequalities* (1.1) *and* (1.2) *and the inequalities*

$$\sin_p \sqrt{rs} \ge \sqrt{\sin_p r \sin_p s} \tag{1.3}$$

and

$$\sinh_p \sqrt{rs} \le \sqrt{\sinh_p r \sinh_p s} \tag{1.4}$$

are all valid.

2. A definition and lemmas

For proving our main results, we need the following definition and lemmas.

Definition 2.1. Let $I \subseteq (0, +\infty)$ be an interval and $f : I \to (0, \infty)$ be a continuous function. This function f is said to be geometrically convex on I if

$$f\big(x^\lambda y^{1-\lambda}\big) \le f^\lambda(x) f^{1-\lambda}(y). \tag{2.1}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$; If the inequality (2.1) is reversed, then the function $f(x)$ is said to be geometrically concave on I.

The notion of geometric convexity was introduced in [9]. For further development on convexity of functions, please refer to [10].

Lemma 2.1 ([10, Proposition 4.3])**.** *Let $f : I \subset (0, \infty) \to (0, \infty)$ be a twice differentiable function. The following assertions are equivalent:*

(1) *The function f is geometrically convex;*
(2) *The function $\frac{xf'(x)}{f(x)}$ is increasing;*
(3) *The function f is geometrically convex if and only if*
$$x\big\{f(x)f''(x) - [f'(x)]^2\big\} + f(x)f'(x) \ge 0 \tag{2.2}$$
holds for all $x \in I$.

Lemma 2.2 ([13])**.** *Let $f : (a, b) \subseteq (0, \infty) \to (0, \infty)$ be a geometrically concave function. Then*

$$g(x) = \int_x^b f(t)\,\mathrm{d}t \quad \text{and} \quad h(x) = \int_a^x f(t)\,\mathrm{d}t$$

are also geometrically concave on (a, b).

Lemma 2.3 ([12])**.** *Let $f : (a, b) \subseteq (0, \infty) \to (0, \infty)$ be monotonic and f^{-1} be the inverse of f. Then*

(1) *if f is increasing and geometrically convex (or geometrically concave respectively), then f^{-1} is geometrically concave (or geometrically concave respectively);*

(2) *if f is decreasing and geometrically convex* (*or geometrically concave respectively*), *then f^{-1} is geometrically convex* (*or geometrically concave respectively*).

3. Proof of Theorem 1.1

We are now in a position to prove our main results.

Proof of Theorem 1.1. Let $f(t)=(1-t^q)^{-1/p}$. Then an easy computation gives

$$\frac{tf'(t)}{f(t)}=\frac{q}{p}\cdot\frac{t^q}{1-t^q} \tag{3.1}$$

and

$$\left(\frac{t^q}{1-t^q}\right)'=\frac{qt^{q-1}}{(1-t^q)^2}. \tag{3.2}$$

From (3.2) we see that the function $\frac{tf'(t)}{f(t)}$ is increasing. By Lemma 2.1, we deduce that the function $f(t)$ is geometrically convex. By Lemma 2.2, we find that the funciton $\arcsin_{p,q}(x)$ is geometrically convex.

On the other hand, let $h(x)=\arcsin_{p,q}(x)$ for $x\in(0,1)$. Then

$$h'(x)=(1-x^q)^{-1/p}$$

which is increasing. By Lemma 2.3, we derive that the function $\sin_{p,q}(x)$ is geometrically concave.

The rest may be proved similarly. □

Proof of Corollary 1.1. This follows from Theorem 1.1 and Definition 2.1. □

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